AGMC August 2024

Individual Round - Senior Group

1. This competition consists of 14 questions, with a total score of 300 points and a duration of 4.5 hours. It includes two sections:

   Section 1: Complete 10 fill-in-the-blank questions from 10:00 to 11:30 AM.
   Section 2: Complete 4 free-response questions from 2:00 to 5:00 PM.

2. Each section's questions are arranged in approximate order of increasing difficulty with the corresponding points marked before each question.

3. Participants are allowed to search online resources and use software tools to assist in answering questions. However, no question requires mandatory use of software tools. Consequently, your written response process must be purely manual.

4. Collaboration or discussing answers with other participants is prohibited.

5. After the competition ends, immediately take photos of your responses and upload them to the designated platform.

Section 1

1. [**8**] Compute the number of real roots of the equation $x^2 = 2024\sin(\pi x)$.

2. [**8**] $f(n)$ is a monotonically increasing function. $f : \mathbb{N}^+ \to \mathbb{N}^+$, and $f(f(n)) = 2n+1$. Compute $f(2024)$.

3. [**10**] Point A is on the x-axis. Lines AB and BC are tangent to the graph of the function $y = x^3$ at points B and C, respectively. If $AB = AC$, compute x_A.

4. **[10]** Let p be a three-digit prime number. The sums of the digits of p and p^2 are equal. Compute the minimum value of p.

5. **[12]** Given that $m, n \in \mathbb{N}^+$, and each a_i is independently chosen as a positive integer less than m. Use explicit expression in terms of m, n to express the probability of satisfying $m | \sum\limits_{i=1}^{n} a_i$.

6. **[12]** Let $\triangle ABC$ be an equilateral triangle with side length 4. Point D is the midpoint of BC. Point P and point B lie on different sides of line AC. $\odot P$ passes through points A and D. Connect BP and CP, and let the $\odot P$ intersect BP at point Q. Compute the maximum distance from point Q to line CP.

7. **[14]** Three positive roots of the equation $ax^3 + bx^2 + cx + d = 0$ are x_1, x_2, x_3, respectively. $a + b = c + d$. Let $f(x) = -\frac{4}{\sqrt{x^2+1}}$, compute the minimun value of

$$\sqrt{3\sqrt{2}f(x_1) + 17} + \sqrt{\sqrt{15}f(x_2) + \frac{31}{2}} + \sqrt{\sqrt{15}f(x_3) + \frac{31}{2}}$$

8. **[14]** Prushka independently and randomly selects n points on the circumference of a circle, then connects each pair of adjacent points to construct an n-sided polygon. Use explicit expression in terms of n to express the probability that all interior angles of this polygon are obtuse.

9. **[16]** For a ten-digit number, if $0 \sim 9$ each appear exactly once in each digit, we call such a number a *Beez number*. A *Beez pair* consists of two *Beez numbers* where one *Beez number* is twice the other *Beez number*. Compute the number of all *Beez pairs*.

10. **[16]** Let $\odot O$ be a circle with a radius of 5, AB and CD are two chords that are mutually perpendicular in $\odot O$, where $AB = 4\sqrt{6}$ and $CD = 8$. Within $\odot O$, there exists a point P such that the circumcircle of $\triangle ACP$ is tangent to the circumcircle of $\triangle BDP$, and the circumcircle of $\triangle BCP$ is tangent to the circumcircle of $\triangle ADP$. Compute OP.

Section 2

11. [**40**] Furina became curious about the mathematical principles behind the music. (Tips: Any music theory or physics-related knowledge is neither required nor allowed when solving this question.) The expression of sound waves is

$$\psi(t) = \sin(2\pi f t)$$

where f is the frequency.

(1) [**20**] The harmonic series consists of a fundamental tone and overtones. If the frequency of the fundamental tone is f, the frequencies of its overtones are $2f$, $3f$, $\ldots$, and nf, respectively. Furina wants to investigate the expression for the superimposed sound wave of the harmonic series with a fundamental frequency of $\frac{1}{2\pi}$Hz, assuming the fundamental tone and its overtones have the same loudness, where n is an integer greater than 1.

① [**8**] Prove:

$$\sum_{k=1}^{n} \sin kx = \frac{\sin \frac{nx}{2} \sin \frac{nx+x}{2}}{\sin \frac{x}{2}}$$

② [**12**] Furina wants to investigate the range of the loudness of the superimposed sound wave of the harmonic series. Let M be the maximum value of $\sum\limits_{k=1}^{n} \sin kx$. Prove:

$$\frac{2}{3}n < M < n$$

(2) [**20**] Furina discovered a continuous function that can roughly simulate the loudness $f(x)$ ($f(x) \neq 0$) through the ratio of the maximum vibration distance of a string to its length x ($-1 \leq x \leq 1$), satisfying the following equation:

$$f(2x^2 - 1) = \frac{f(x)}{f(x) + 1}$$

Construct one of the functions $f(x)$ satisfying the above conditions.

12. [**40**] When Number Theory falls in love with Geometry...

Minkowski Theorem is an effective tool connecting number theory and geometry.

In the Euclidean space R^n, a bounded centrally symmetric convex body A that satisfies $\mathrm{Vol}(A) > 2^n$ must contain a lattice point different from the origin, where a lattice point refers to a point where all coordinates are integers and n refers to the dimension. For example, when $n = 2$, the theorem's weaker version states that in the Euclidean plane, any origin-centered symmetric convex closed region with area exceeding 4 must contain a non-origin lattice point.

(1) [**8**] In a three-dimensional space, every lattice point except the origin is the center of a solid carbon nanotube black body sphere with a radius of r. A laser is emitted from the origin, and it will be absorbed when it hits a solid carbon nanotube black body sphere. Prove that any laser will travel a distance of no more than $\frac{\mathrm{e}}{2r}$ before being absorbed.

(2) [**12**] Let $n \in \mathbb{N}^+$. Prove: If the equation

$$x^2 + xy + y^2 = n$$

has rational solutions, it has positive integer solutions.

(3) [**20**] For each $n \in \mathbb{N}^+$, let $f(n)$ denote the number of methods to express n as a sum of non-negative integer powers of 2. Representations that differ only in the order, such as $2^1 + 2^2$ and $2^2 + 2^1$, are considered the same method. For example, since

$$4 = 2^2 = 2^1 + 2^1 = 2^1 + 2^0 + 2^0 = 2^0 + 2^0 + 2^0 + 2^0,$$

we have $f(4) = 4$.

Prove that there exist real numbers a, b, c_1, c_2, such that

$$c_1 n^2 - c_2 n \ln n - an < \ln f(2^n) < c_1 n^2 - c_2 n \ln n - bn,$$

and solve for c_1, c_2.

13. **[50]** After learning Music Theory, Furina plans to learn to play a traditional musical instrument to develop practical musical ability.

Orchestration's Prelude

In music composition, *Retrograde* and *Inversion* are common techniques used for transforming melodies: *Retrograde* refers to playing a melody backward, creating a left-to-right symmetrical mirror effect on the melody; *Inversion* refers to flipping a melody upside down, creating an up-and-down symmetrical mirror effect on the melody.

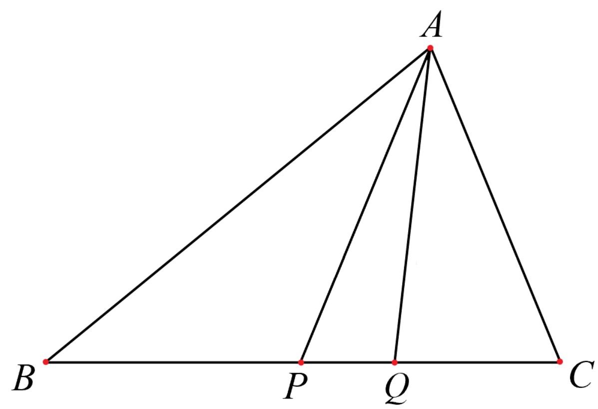
The application of these two techniques on a music composition epitomizes the concept of two *Isogonal Lines* in geometry world. Given an acute triangle $\triangle ABC$, if there exist points P and Q on side BC such that $\angle BAP = \angle CAQ$, then AP and AQ are called the *Isogonal Lines* of $\triangle ABC$. (We will only discuss *Isogonal Lines* in acute triangles for convenience later)

The *Balalaika*, a type of string instrument from Russia with a triangular body and three strings, radiates the brilliance of geometric elements from its structure, sound projection, and overall design to the mathematical precision of its string arrangements. With its three strings and distinctive geometric appearance, this enchanting instrument captures the soul of Russian folk traditions while serving as a quintessential cultural ambassador that continues to charm audiences worldwide with its melodious voice, aesthetic design, and expressive versatility.

I am in need of music that would flow

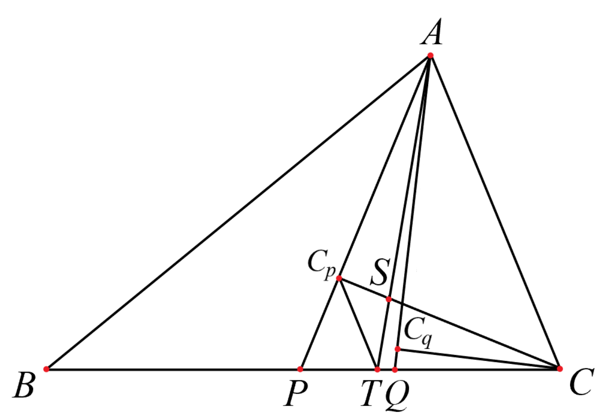
[**5**] Assuming that the strings have equal amplitudes when they vibrate, which means the two lines are *Isogonal Lines*, we abstract this situation into the geometric model mentioned before. Prove:

$$\frac{AB^2}{AC^2} = \frac{BP \cdot BQ}{CP \cdot CQ}$$



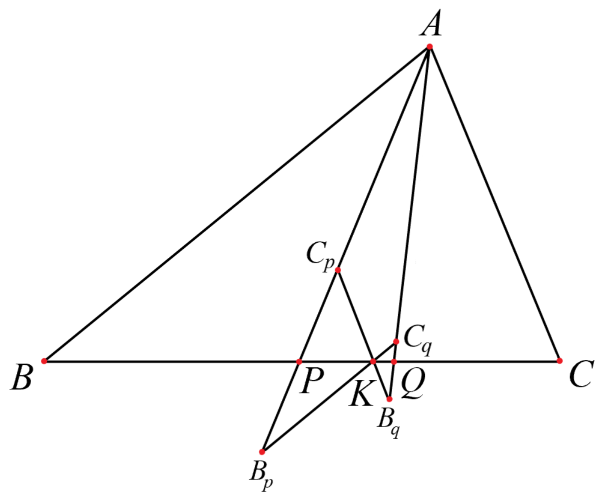
Over my fretful, feeling fingertips,

[**10**] The optimal angle for plucking the string is when it is perpendicular. There is a method to find a sound wave circle: C_p, C_q are projections of point C on AP, AQ, respectively. Construct $TC_p//AC$ and let it intersect BC at point T. Line AT intersects CC_p at point S. If P is the midpoint of BC, prove that points P, S, C_p, C_q lie on the same sound wave circle, or in other words, they are concyclic.



Over my bitter-tainted, trembling lips,

[**15**] *Balalaika* has different ranges and pitches due to its different sizes and dimensions. It is comparable to a symphony orchestra when played together in multiple numbers. Even so, all such instruments follow the following rule: the projections of point B on strings AP and AQ are B_p and B_q, respectively. Connect B_pC_q, B_qC_p, and let them intersect at point K. Prove that point K lies on line BC and find out the range of the trajectory of point K on line BC.



With melody, deep, clear, and liquid-slow.

[**20**] The resonance chamber of *Balalaika* is also a triangle, with points D, E, and F on sides AB, BC, and AC, respectively. Thus, $\triangle DEF$ is the resonance chamber. To achieve the optimal acoustic quality, the incenter of $\triangle ABC$ and the centroid of $\triangle DEF$ need to coincide. Prove that $S_{\triangle DEF}$ reaches its minimum value if and only if $AD + BE + CF = AF + BD + CE$, and use an explicit expression in terms of a, b, and c to express the minimum value of $S_{\triangle DEF}$, where $BC = a$, $AC = b$, and $AB = c$.

14. **[50]** Math top student Albert and English top student Barbara are not honest. They are going to take an English exam, which consists entirely of multiple-choice questions. They discussed beforehand that Barbara would send signals to Albert with the options during the exam. However, all the questions turned out to be questions with an undetermined number of options. It is known that Albert cannot answer any of the English questions but is very proficient in analysis. Albert knows that Barbara has a habit of writing the options for each question in alphabetical order like A $\to$ B $\to$ C $\to$ D. Assume Barbara can always answer the English questions correctly.

For example, suppose the exam has three questions, each with options A, B, and C, and Barbara sent four choice signals to Albert. If the options Barbara sent are "ACBB", Albert can deduce that the answers must be (A)(C)(B)(B). However, if the options Barbara sent are "ABCC", Albert cannot determine whether the answers are (A)(BC)(C) or (AB)(C)(C).

Assume the exam consists of m questions, and Barbara sent n choice signals to Albert, where $m, n \in \mathbb{N}^+$.

(1) **[25]** Assume that each question has only two options: A and B, with the number of correct options may be either 1 or 2.

① **[1]** Prove: $m \leq n \leq 2m$.

② **[24]** Use an explicit expression in terms of m and n to express the number of arrangements of options that will definitely enable Albert to achieve full marks in the exam.

(2) **[25]** Assume that each question has four options: A, B, C, and D, with the number of correct options possibly being 1, 2, 3, or 4. Use an explicit expression in terms of m and n to express the number of arrangements of options that will definitely enable Albert to achieve full marks in the exam.